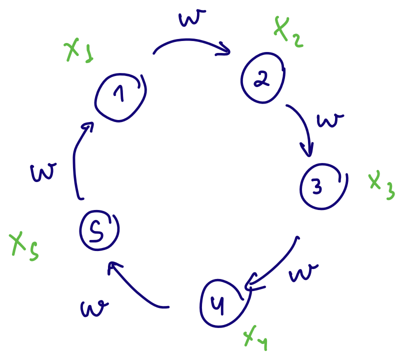
Lecture 2 EN.553.744 Prof. Luana Ruiz 1/27

Today:
- interpretation of graph Laplacian
- TV of graph signals
- graph frequencies & oscillation modes
- Graph Fourier Transform
- graph convolutions

E.g.: Interpretation of the (left) graph Laplacian.



5-cycle digraph

"5-sample discrete-time graph"

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & w \\ w & 0 & 0 & 0 & 0 \\ 0 & w & 0 & 0 & 0 \\ 0 & 0 & w & 0 & 0 \\ 0 & 0 & 0 & w & 0 \end{bmatrix}$$

$$A_{ij} = \begin{cases} w(j,i) & \text{if } (i,j) \in \mathcal{E} \\ 0 & \text{o.w.} \end{cases}$$

(left) Laplacian : $L = D_{out} - A = \mathrm{diag}(A.\mathbb{1}) - A$

$$= \begin{bmatrix} w & 0 & 0 & 0 & -w \\ -w & w & 0 & 0 & 0 \\ 0 & -w & w & 0 & 0 \\ 0 & 0 & -w & w & 0 \\ 0 & 0 & 0 & -w & w \end{bmatrix}$$

Let $w = \frac{1}{\Delta t}$

<u>and</u> $x = [x_1 \; x_2 \; x_3 \; x_4 \; x_5]$ sampled with period Δt

<u>and</u> $z = Lx$. Then, $z_i = \begin{cases} \frac{1}{\Delta t}(x_i - x_{i-1}) & 2 \leq i \leq 5 \\ \frac{1}{\Delta t}(x_1 - x_5) & i = 1 \end{cases}$

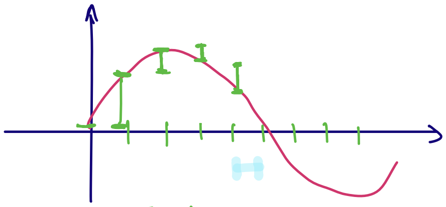
=> the graph laplacian generalizes differentiation to arbitrary graphs

E.g.: Interpretation of the normalized symmetric Laplacian

Consider once again the 5-sample discrete time graph / signal.

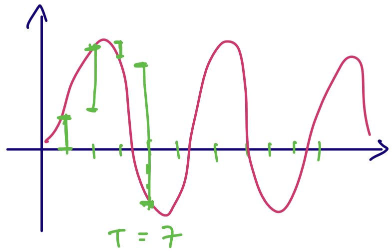
In DSP, the total variation energy of DT signals is defined as : $TV(x) = \sum_i |x_i - x_{i-1}|^2$

It is a proxy for the signal frequency (and in fact can be used to estimate it)



T = 10
F ≈ 1/10

lower TV



T = 7
F ≈ 1/7

higher TV

Let $w = 1$ (unweighted G) wlog.

symmetric Laplacian

Claim: $TV(x) = x^T L x = \langle x, x \rangle_2$

Proof:

$$A_{sym} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \Rightarrow L = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix}$$

$$x^T L x = (x_1 \; x_2 \; x_3 \; x_4 \; x_5) \overbrace{\begin{pmatrix} 2x_1 - x_2 - x_5 \\ 2x_2 - x_1 - x_3 \\ 2x_3 - x_2 - x_4 \\ 2x_4 - x_3 - x_5 \\ 2x_5 - x_4 - x_1 \end{pmatrix}}^{Lx}$$

$$\begin{aligned} = \; & x_1((x_1 - x_5) - (x_2 - x_1)) \\ & + x_2((x_2 - x_1) - (x_3 - x_2)) \\ & + x_3((x_3 - x_2) - (x_4 - x_3)) \\ & + x_4((x_4 - x_3) - (x_5 - x_4)) \\ & + x_5((x_5 - x_4) - (x_1 - x_5)) \end{aligned}$$

$$= (x_1 - x_5)^2 + (x_2 - x_1)^2 + (x_3 - x_2)^2 + (x_4 - x_3)^2 + (x_5 - x_4)^2$$

$$= \sum_i (x_i - x_{i-1})^2$$

$\Rightarrow$ $TV(x) = x^T L x$ generalizes the notion of total variation energy (and thus of signal frequency) to arbitrary graphs

The Graph Fourier Transform (GFT)

Recall $TV(x) = x^T L x$. Assuming $\|x\|_2 = 1$,

what are the lowest & highest values $TV(x)$ can take?

↳ Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of L, ordered by increasing value

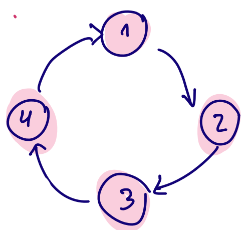
Since $L.\mathbb{1} = 0$ and L is PSD, → Ex.: prove

$\lambda_1 = 0$, $\lambda_2, \lambda_3, \ldots, \lambda_n$

Then, $\begin{cases} \max\limits_{\|x\|=1} TV(x) = \lambda_n \\ \min\limits_{\|x\|=1} TV(x) = 0 \end{cases}$

The Laplacian eigenvalues are the graph's canonical frequencies;
the eig. vectors are corresponding oscillation modes

E.g.:



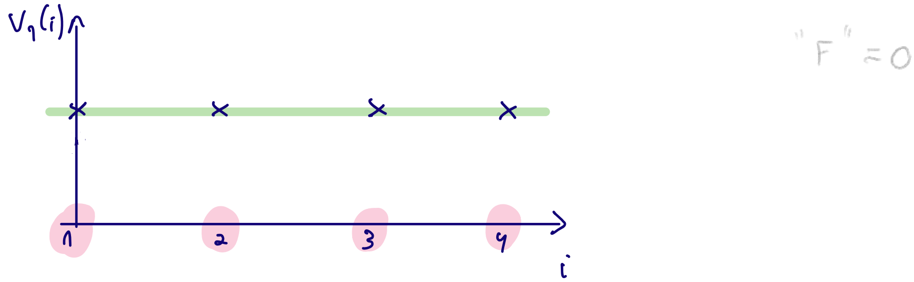
always assumed symmetric from now on

$$L = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$ hence, diagonalizable

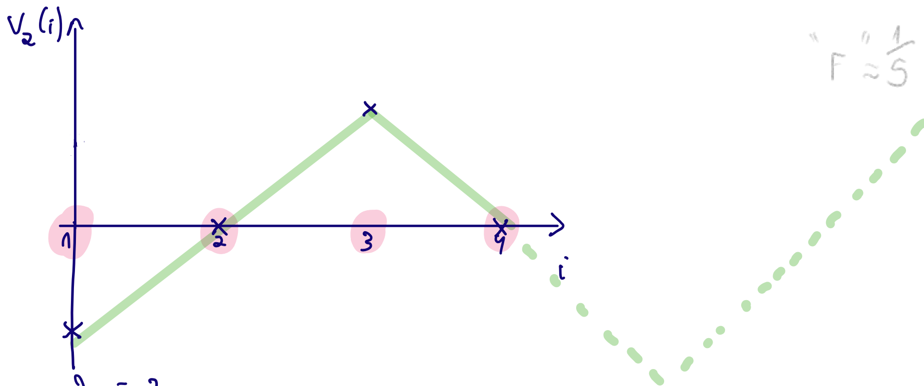
$\lambda_1 = 0$ $\quad v_1 = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}$ $\qquad \lambda_2 = 2$ $\quad v_2 = \begin{bmatrix} -\sqrt{2}/2 \\ 0 \\ +\sqrt{2}/2 \\ 0 \end{bmatrix}$

$\lambda_3 = 2$ $\quad v_3 = \begin{bmatrix} 0 \\ -\sqrt{2}/2 \\ 0 \\ +\sqrt{2}/2 \end{bmatrix}$ $\qquad \lambda_4 = 4$ $\quad v_4 = \begin{bmatrix} -0.5 \\ 0.5 \\ -0.5 \\ 0.5 \end{bmatrix}$

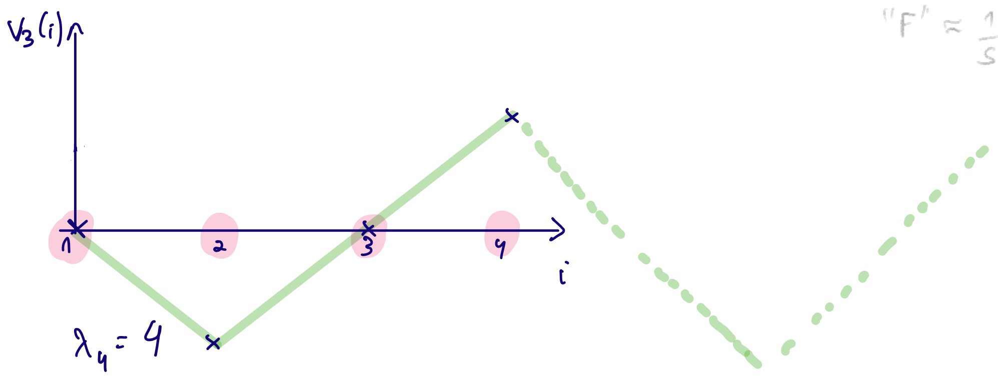
$\lambda_1 = 0$



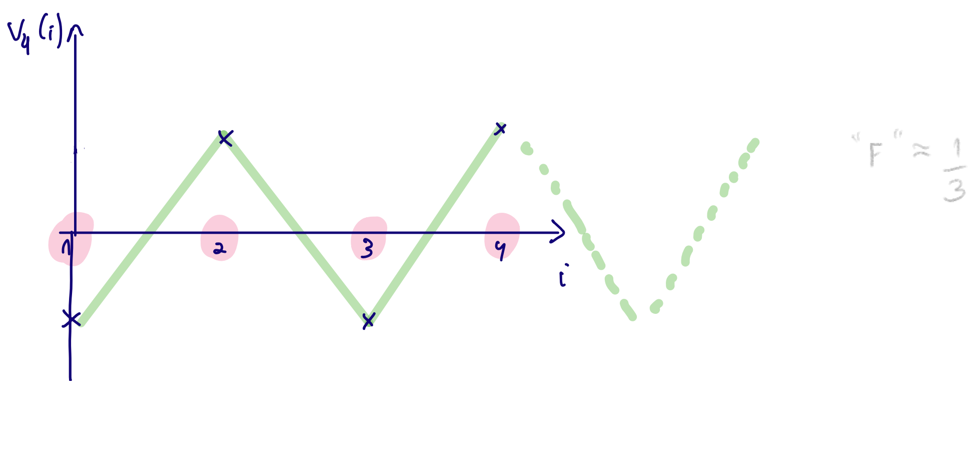
$\lambda_2 = 2$



$\lambda_3 = 2$



$\lambda_4 = 4$



Since in $L = V \Lambda V^H$ V is orthonormal, its columns — the eigenvectors or oscillation modes — form a basis of $\mathbb{R}^n$.

This means we can represent any graph signal x in this basis

More generally, this is true not just for the Laplacian (which is symmetric and thus always diagonalizable) but for any diagonalizable GSO S.

E.g.: adjacencies of undirected graphs
adjacencies of directed graphs / RW adjacencies & Lapl. w/ $\neq$ eigs
Laplacians w/ $\neq$ eigs.

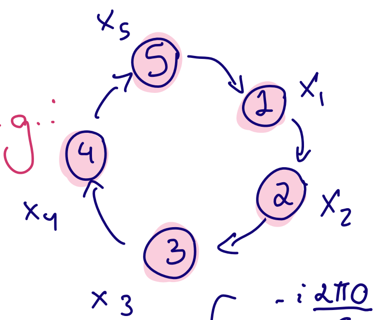
Obs.: Even if the S is not diagonalizable, we can still express graph signals on its generalized eigenvector basis (from the Jordan normal form)

(DEF) Given a diagonalizable GSO $S = V \Lambda V^H$ and a signal x, the projection of x onto V is called x's graph Fourier transform:

$$GFT(x) = \hat{x} = V^H x \qquad \hat{x} \in \mathbb{R}^n$$

$$[\hat{x}]_i = v_i^H x = \langle v_i, x \rangle$$

E.g.:



$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} ; \quad N = 5$$

$$\Lambda = \begin{bmatrix} e^{-\frac{i2\pi 0}{5}} & & & & \\ & e^{-\frac{i2\pi\cdot 1}{5}} & & \underline{0} & \\ & & e^{-\frac{i2\pi\cdot 2}{5}} & & \\ & \underline{0} & & e^{-\frac{i2\pi\cdot 3}{5}} & \\ & & & & e^{-\frac{i2\pi\cdot 4}{5}} \end{bmatrix}$$

$n \quad k$

$$V = \frac{1}{\sqrt{5}} \begin{bmatrix} e^{-\frac{i\cdot 2\pi\cdot 0\, 0}{5}} & e^{-\frac{i\cdot 2\pi\cdot 0\cdot 1}{5}} & \\ e^{-\frac{i\cdot 2\pi\cdot 1\cdot 0}{5}} & e^{-\frac{i\cdot 2\pi\cdot 1\cdot 1}{5}} & \\ e^{-\frac{i\cdot 2\pi\cdot 2\cdot 0}{5}} & \vdots & \\ e^{-\frac{i\cdot 2\pi\cdot 3\cdot 0}{5}} & & \\ e^{-\frac{i2\pi\cdot 4\cdot 0}{5}} & & \end{bmatrix}$$

The GFT of x is then:

$$[\hat{x}]_k = \frac{1}{\sqrt{5}} \sum_{n=0}^{4} x_{n+1}\, e^{-\frac{i2\pi nk}{5}}$$

$$= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x_{n+1}\, e^{-\frac{i2\pi nk}{N}}$$

→ Discrete FT
↳ look it up!

Ex.: show the above step-by-step

→ The GFT generalizes the DFT in Euclidean space to the graph signal space

<u>Obs.</u>: In general, the interpretation of the adjacency eigenvalues as graph frequencies is not as clean as it is for the Laplacian eigenvalues (eigenvectors of A & L don't match in general, and there is no alternative TV definition in terms of A).

But in practice it is generally true that low magnitude adjacency eigenvalues correspond to high graph frequencies, and vice-versa.

When considering the normalized versions of the adjacency & Laplacian respectively, the eigenvectors are the same, but the adjacency eigenvalues should NOT be interpreted as graph frequencies.

$$\bar{A} = D^{-1/2} A \cdot D^{-1/2} \rightarrow \bar{L} = I - \bar{A} = I - D^{-1/2} A D^{-1/2}$$