Lecture 17 EN.553.744 Prof. Luana Ruiz

→ Does the GFT converge to the WFT?

Why do we care?

high variance of small samples might obscure important information in a signal's GFT, such as correlation wrt graph structure (see colab)

↳ graphon limit devoid of noise, reveals real, intrinsic signal structure.

-) convergent graph signals $(G_n, x_n) \to (W, \mathcal{X})$
-) Eigenvalue convergence: $\frac{\lambda_i(G_n)}{n} \to \lambda_i(W) \; \forall i$

Conjecture 1: the GFT $[\hat{x}_n]_i$ converges to the WFT $[\hat{\mathcal{X}}]_i \; \forall i$

Recall $[\hat{x}_n]_i = \underline{\langle x_n, v_i^n \rangle}$; $[\hat{\mathcal{X}}]_i = \underline{\langle \mathcal{X}, \varphi_i \rangle}$

In order to have convergence GFT → WFT, we need the graph eigenvectors v_i^n to converge to the graphon eig-fcns φ_i

(THM) Davis-Kahan

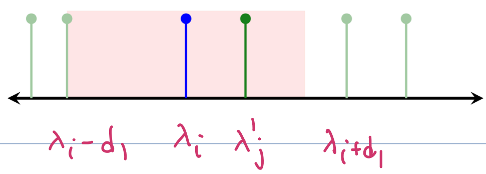
Let T and T' be self-adjoint HS operators w/ eigenspectra (λ_i, φ_i) & (λ_i', φ_i') resp., ordered by eigenvalue magnitude.

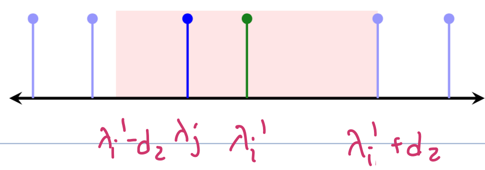
Then,

$$\|\varphi_i - \varphi_i'\| \leq \frac{\pi}{2} \frac{\|T - T'\|}{d(\lambda_i, \lambda_i')}$$

where $d(\lambda_i, \lambda_i')$ is defined as:

$$d(\lambda_i, \lambda_i') = \min\Big(\underbrace{\min_{j \neq i} |\lambda_i - \lambda_j'|}_{d_1}, \underbrace{\min_{j \neq i} |\lambda_i' - \lambda_j|}_{d_2} \Big)$$





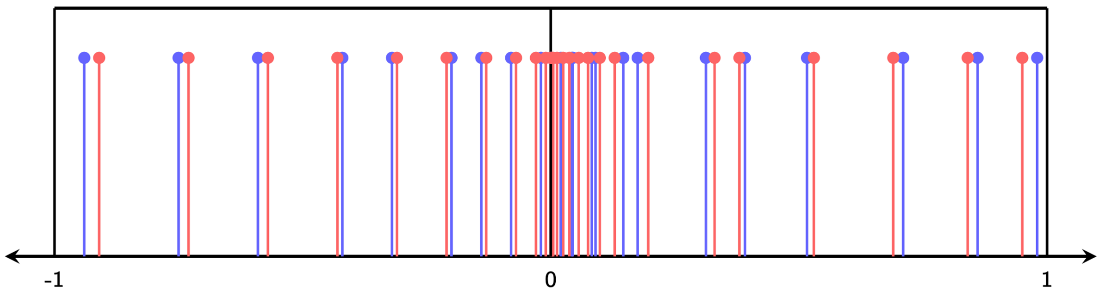
So we can apply DK theorem to T_W & T_{W_n} (graphon W_n is induced by G_n):

$$\| \varphi_i(T_{W_n}) - \varphi_i(T_W) \| \leq \frac{\pi}{2} \frac{\| T_W - T_{W_n} \|}{d(\lambda_i(T_W), \lambda_i(T_{W_n}))}$$

From Lec. 15, we know that if G_n converges to W in the cut norm, T_{W_n} converges to T_W in the L_2 operator norm ($\|W\|_{2\to 2} = \|T_W\|_2$). Hence, the numerator converges.

So we are good, right? No. The denominator might vanish as $i \to \infty$.

Though the eigenvalues converge, their convergence is not uniform because the graphon eigenvalues accumulate at 0.

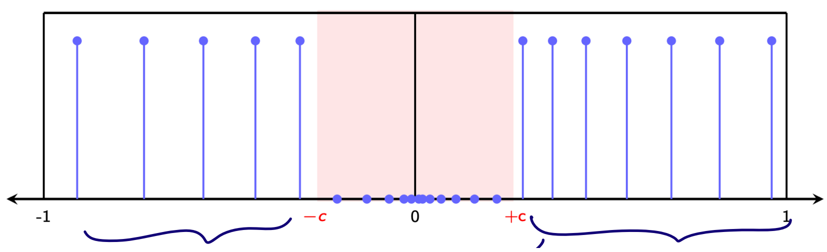


Convergence holds in the sense that $\exists\, n_0$ s.t $\forall\, n > n_0$, $\left| \frac{\lambda_i(G_n)}{n} - \lambda_i(W) \right| < \epsilon$, $\epsilon > 0$.

However, n_0 will be different for each i.

Conjecture 1 is wrong, but not so far off.

(DEF) A graphon signal (W, x) is c-bandlimited if its WFT satisfies $[\hat{x}]_i = 0$ for all $i \in \mathcal{C} = \{ j \text{ s.t. } |\lambda_j| > c \}$



— $\mathcal{C}$ —

Conjecture 2: Let (W, x) be c-bandlimited & (G_n, x_n) a sequence converging to (W, x). Then, the GFT $[\hat{x}_n]_i$ converges to the WFT $[\hat{x}]_i$.

Pf.: $|[\hat{x}_n]_i - [\hat{x}]_i| = |\langle x_n, \varphi_i^n \rangle - \langle x, \varphi_i \rangle|$

$= |\langle x_n, \varphi_i^n \rangle \pm \langle x, \varphi_i^n \rangle - \langle x, \varphi_i \rangle|$

$\underset{\text{C.S. \& } \Delta \text{ ineq.}}{\leq} \|x - x_n\| \|\varphi_i^n\| + \|x\| \|\varphi_i^n - \varphi_i\|$ for $i \in \mathcal{C}$

→ By convergence of x_n, $\exists\, n_1$, s.t. $\|x_n - x\| \leq \frac{\epsilon}{2}$ $\forall\ n > n_1$ and $\epsilon > 0$.

→ As for $\|\varphi_i^n - \varphi_i\|$, we have:

$(\forall i)$

$$\|\varphi_i^n - \varphi_i\| \leq \frac{\pi}{2} \frac{\|T_W - T_{W_n}\|}{d(\lambda_i, \lambda_i^n)} \leq \frac{\pi}{2} \frac{\|T_W - T_{W_n}\|}{\min\limits_{i \in \mathcal{C}} d(\lambda_i, \lambda_i^n)}$$

and from $T_{W_n} \to T_W$, $\exists\ n_2$ s.t. $\|\varphi_i^n - \varphi_i\| \leq \epsilon / 2\|x\|$

$\forall \; n > n_2$

Therefore, $\forall \; n > \max\{n_1, n_2\}$ & $\forall \; i \in \mathcal{C}$,

$$|[\hat{x}_n]_i - [\hat{x}]_i| \leqslant \|x_n - x\| \cancel{\|\varphi_i^n\|}^{1} + \|x\|\,\|\varphi_i^n - \varphi_i\|$$

$$\leqslant \frac{\varepsilon}{2} + \cancel{\|x\|} \cdot \frac{\varepsilon}{2\cancel{\|x\|}} = \varepsilon$$

For $j \notin \mathcal{C}$, we have:

$$\langle \varphi_j(T_{W_n}), x_n \rangle \longrightarrow \langle \underbrace{\psi}_{\psi \in \perp \text{span}\{\varphi_i;\, i \in \mathcal{C}\}}, x \rangle = 0 = \langle \varphi_j(T_W), x \rangle$$

$\square$

→ Graphon convolutions

Graphon filters have the same algebraic structure of graph filters: shift-and-sum operations where shift is now the graphon shift T_W

(DEF) Given a graphon signal (W, x), we write the graph convolution as the map:

$$T_H : L_2([0,1]) \to L_2([0,1])$$
$$T_H : x \mapsto y \quad \text{with}$$

$$\boxed{y = T_H x = \sum_{k=0}^{K-1} h_k \, T_W^{(k)} x} \quad \text{and}$$

$$T_W^{(k)} x = \begin{cases} \int_0^1 W(u, \cdot)\left(T_W^{(k-1)} x\right)(u)\, du, & k \geq 1 \\ I \text{ (identity)}, & k = 0 \end{cases}$$

Like graph convolutions, graphon convolutions also admit a spectral representation.

(THM) Given a graphon convolution with input x & output y, the WFTs $\hat{x}$ of $\hat{y}$ are related as:

$$\hat{y}_j = \underbrace{\sum_{k=0}^{K-1} h_k \lambda_j^k}\, \hat{x}_j = \hat{T}_H(\lambda_j) \cdot \hat{x}_j = \underbrace{h(\lambda_j)}\, \hat{x}_j$$

Pf.:

$$y = \sum_{k=0}^{K-1} h_k \sum_{j \in \mathbb{Z}\setminus\{0\}} \lambda_j^k \varphi_j(v) \underbrace{\int_0^1 \varphi_j(u) x(u)\, du}_{\hat{x}_j}$$

$$y = \sum_{k=0}^{K-1} h_k \sum_{j \in \mathbb{Z}\setminus\{0\}} \lambda_j^k \varphi_j \hat{x}_j$$

$$\hat{y}_i = \int_0^1 y(u) \varphi_i(u)\, du = \sum_{k=0}^{K-1} h_k \lambda_i^k \cancel{\int \varphi_i^2(u)\, du}^{\,1} \hat{x}_i$$

$$\Rightarrow \hat{y}_i = \sum_{k=0}^{K-1} h_k \lambda_i^k \hat{x}_i \quad \square$$

Simple result following from diagonalization of $T_W^{(k)}$ by φ_j. But interesting takeaways:

- like WFT, spectral response of graphon convolution is discrete
- it is also pointwise; the j^{th} spectral comp. of y only depends on λ_j & $\hat{x}_j$

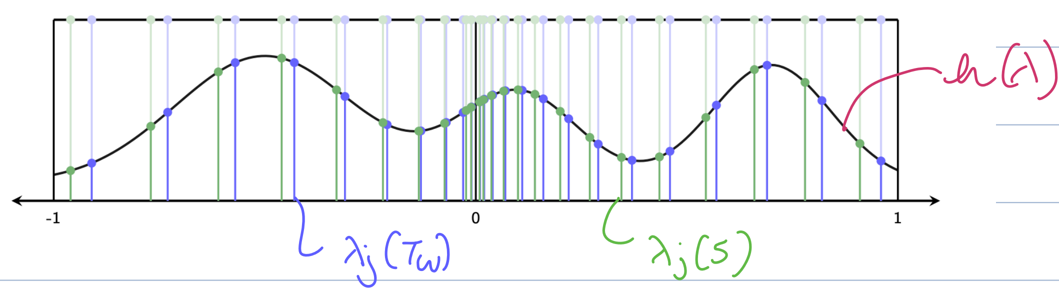
- the spectral response of T_H, $h(\lambda) = \sum_{k=0}^{K-1} h_k \lambda^k$, is $\perp\!\!\!\perp$ of W (like the spectral resp. of $H(S)$, $h(\lambda)$, was $\perp\!\!\!\perp$ of the graph)

- the graphon only determines where we evaluate this function $\rightarrow$ at the eigvals $\{\lambda_i\}_i$.

And most importantly: given the same coeffs. h_k, define the graph convolution $H(S)x = \sum_{k=0}^{K-1} h_k S^k x$. Its spectral response is $h(\lambda) = \sum_{k=0}^{K-1} h_k \lambda^k$

the same as the spectral resp. of the graphon convolution!

(except for where they are evaluated:
↳ at the graphon eigs. $\lambda_j(T_W)$ for T_H;
↳ at the graph eigs. $\lambda_j(S)$ for $H(S)$)



We know:

1) Given a convergent sequence $G_n \to W$, we know
$\frac{\lambda_i(G_n)}{n} \to \lambda_i(W) \quad \forall\, i$

2) If we fix coeffs. h_k, the graph conv. $H(S)$ & the graphon conv. T_H have the same spectral response
$$h(\lambda) = \sum_{k=0}^{K-1} h_k \lambda^k$$

So?

(THM) Given $G_n \to W$ & $H(G_n) = \sum_{k=0}^{K-1} h_k \left(\frac{A_n}{n}\right)^k$ and
$T_H = \sum_{k=0}^{K-1} h_k T_W^{(k)}$, we have:

$$\hat{H} \longrightarrow \hat{T}_H \quad \text{in the sense that}$$
$$\hat{T}_H(\lambda_j(T_{W_n})) \to \hat{T}_H(\lambda_j(T_W)) \quad \forall\, j \in \mathbb{Z} \setminus \{0\}$$

i.e., the graph convolution converges to the graphon convolution in the spectral domain

Yet: spectral convergence is not enough; graph convs. operate in node domain.