Lecture 16 EN.553.744 Prof. Luana Ruiz

▶ Graphon signal processing

A graphon signal is defined as a function

$$x : [0,1] \to \mathbb{R} \qquad (x \in \mathbb{R}^n)$$

We focus on signals in L_2, $x \in L_2([0,1])$:

$$\int_0^1 |x(u)|^2 du \leq B < +\infty$$

"Finite energy signals"

Like a graphon, graphon signals are limits of convergent sequences of graph signals

Induced graphon signals: Let (G_n, x_n) be a graph signal. The induced graphon signal is defined as:

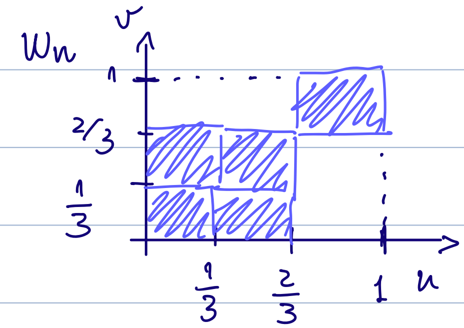
W_n is as in (*) in Lec. 15

$$x_n(u) = \sum_{j=1}^{n} [x_n]_j \, \mathbb{I}(u \in I_j)$$

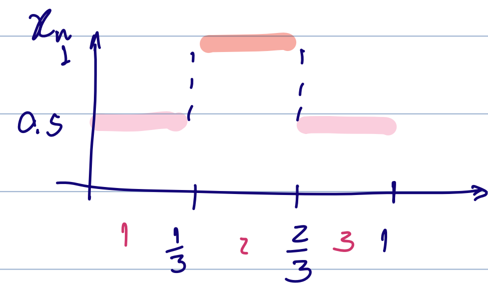
$\mathbb{I}$ is indicator fcn; $\quad I_j = \begin{cases} \left[\frac{j-1}{n}, \frac{j}{n}\right), & 1 \le j \le n-1 \\ \left[\frac{n-1}{n}, 1\right], & j = n \end{cases}$

E.g.:

$$A_n = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ 3 \\ 2 \\ 1 \end{array}$$



$$x_n = \begin{bmatrix} 0.5 \\ 1 \\ 0.5 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$



Convergent sequences of graph signals:

A sequence of graph signals (G_n, x_n) converges to the graphon signal (W, X) if $\exists$ a sequence of permutations $\{\pi_n\}_n$ such that for all motifs F,

$$t(F, G_n) \longrightarrow t(F, W) \quad \&$$

$$\| x_{\pi_n(G_n)} - x \|_2 \to 0$$

We write $\{(G_n, x_n)\}_n \to (W, x)$

↳ the permutation seq. is independent of node labels

(we could do w/o them completely by defining a cut distance for graphon signals; this is done in Levie, 2023. However, since we are operating w/ signals in L_2, we will stick with this defn. for now)

→ The graphon (shift) operator

The graphon W can be used to define an integral linear operator

$$T_W : L^2([0,1]) \to L^2([0,1])$$
$$x \mapsto T_W x$$

$$\boxed{T_W x = \int_0^1 W(u, \cdot)\, x(u)\, du}$$

This is a Hilbert-Schmidt operator:

if is continuous & compact

mapping bounded sets to subsets with compact closure

It is a HS operator because W is bounded, so it is in L_2

It has HS norm $\|T_W\|_{HS}^2 = \|W\|_2^2$

$= \int_0^1 \int_0^1 W^2(u,v)\,du\,dv$

T_W is called the "graphon shift operator because it "diffuses" graphon signals over the graphon

→ The graphon Fourier transform (WFT)

Besides being a HS operator (and thus compact), T_W is self adjoint due to symmetry of W:

$$\langle T_W x, y \rangle = \langle x, T_W y \rangle$$

Spectral theorem for self-adjoint compact operators on Hilbert spaces H:

For every such T, $\exists$ an orthonormal basis of H consisting of eigenfunctions of T. This basis, $\{\varphi_i\}_i$, is countably infinite, with corresponding real eigenvalues $\{\lambda_i\}_i$, satisfying $\lambda_i \to 0$.

In our case, $T = T_w$ and $H = L_2([0,1])$

The function $\varphi : [0,1] \to \mathbb{R}$ is an eigenfcn of T_w with eigenvalue λ if:

$$T_w \varphi = \lambda . \varphi \qquad (Sv = \lambda v)$$

There are infinitely many such λ, φ pairs (possibly w/ geometric multiplicity larger than one), but the eigenpairs are countable

$$\{(\lambda_i, \varphi_i)\}_{i=1}^{\infty}$$

Since the φ_i form an orthonormal, they have unit norm $\|\varphi_i\|^2 = \int_0^1\int_0^1 \varphi_i^2(u)\, du$

→ We can write the graphon W in the basis $\{\varphi_i\}_{i=1}^{\infty}$ as:

$$\boxed{W(u,v) = \sum_{i=0}^{\infty} \lambda_i \varphi_i(u) \varphi_i(v)}$$

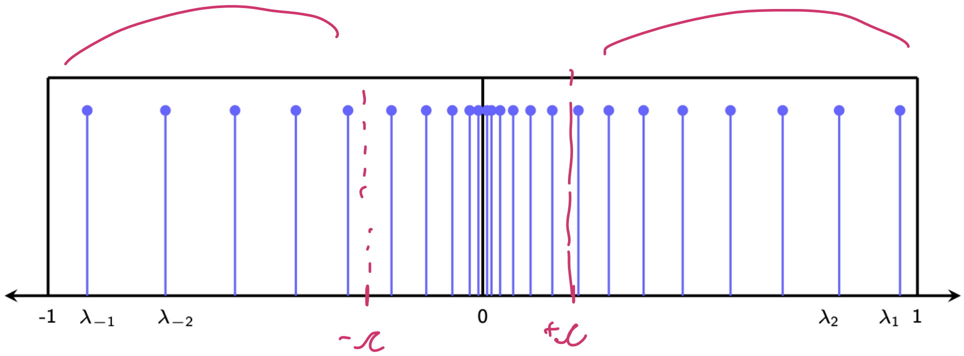
$(S = V \Lambda V^H)$

Eigenvalue range: since $0 \leq W \leq 1$, $\|T_W\|_2 \leq 1$, which means the eigenvalues of T_W lie in $[-1, 1]$. $\|T_W\|_{HS} \leq 1$, the only possible accumulation point is 0 (c.f. spectral theorem)

We will leverage this to order the eigenvalues as $\{\lambda_j\}_{j \in \mathbb{Z} \setminus \{0\}}$ with:

$$\lambda_{-1} \leq \lambda_{-2} \leq \ldots \leq 0 \leq \ldots \leq \lambda_2 \leq \lambda_1$$

E.g.:



eigenvalue accumulation at 0 (and only at 0) also means that, for any $c > 0$,

$$\#\{\lambda_i : |\lambda_i| \geq c\} = n_c < +\infty$$

Eigenvalue convergence:

(THM) Let $(G_n)_n$ be a sequence converging to a graphon W. Then,

adjacency

$$\lim_{n\to\infty} \frac{\lambda_j(A_n)}{n} = \lambda_j(W) = \lim_{n\to\infty} \lambda_j(T_{W_n})$$

graphon induced by G_n (A_n)

Pf. sketch:

$$W_n(u,v) = \sum_{i,j=1}^{n} [A_n]_{ij} \, \mathcal{I}(u \in I_i) \, \mathcal{I}(v \in I_j)$$

& $A_n = V_n \Omega_n V_n^T$; therefore:

$$T_{W_n} \varphi(v) = \int_0^1 W_n(u,v) \varphi(u) \, du = \lambda \varphi(v)$$

$$= \int_0^1 \sum_{i,j=1}^{n} [A_n]_{ij} \, \mathcal{I}(u \in I_i) \, \mathcal{I}(v \in I_j) \, \varphi(u) \, du = \lambda \varphi(v)$$

$$\left(\int_0^1 \sum_{i=1}^{n} [A_n]_{ij} \, \mathcal{I}(u \in I_i) \, \varphi(u) \, du \right) = \lambda \varphi(v \in I_j)$$

$$\left(\sum_{i=1}^{n} [A_n]_{ij} \int_0^1 \mathcal{I}(u \in I_i) \, \varphi(u) \, du \right) = \lambda \varphi(v \in I_j)$$

$$(**) \quad \lambda \varphi(v \in I_j) = \underbrace{\sum_{i=1}^{n} [A_n]_{ij} \int_0^1 \mathcal{I}(u \in I_i) \, \varphi(u) \, du}_{\text{constant } \forall v \in I_j}$$

$\Rightarrow$ φ is a stepfunction $\varphi(v \in I_j) = x_j$

Rewrite $(**)$ as: $\lambda x_j = \sum_{i=1}^{n} [A_n]_{ij} \, x_i \, \frac{1}{n}$

$\Rightarrow \lambda x = \frac{1}{n} A_n x \Rightarrow \lambda_j(T_{W_n}) = \frac{\lambda_j(A_n)}{n} \quad \forall j$

For convergence: •) $t(C_k, G_n) = \sum_{i=1}^{n} \lambda_i^k$

•) $t(C_k, G_n) \longrightarrow t(C_k, W)$

We are finally ready to define the graphon Fourier transform.

Note that $T_W \chi$ can be written as:

$$(T_W \chi)(v) = \int_0^1 W(u,v)\,\chi(u)\,du$$

$$= \int_0^1 \sum_{j \in \mathbb{Z} \setminus \{0\}} \lambda_j \varphi_j(u)\,\varphi_j(v)\,\chi(u)\,du$$

$$= \sum_{j \in \mathbb{Z} \setminus \{0\}} \lambda_j \varphi_j(v) \int \chi(u)\varphi_j(u)\,du$$

$$= \sum_{j \in \mathbb{Z} \setminus \{0\}} \lambda_j \varphi_j(v) \langle x, \varphi_j \rangle$$

↳ I.e., we can represent the signal x in the graphon eigenbasis

The change of basis is the graphon Fourier transform.

(DEF) The graphon FT of a graphon signal (W, x) is a functional $\hat{x} = \text{WFT}(x)$

$$\boxed{\hat{x}_j = \hat{x}(\lambda_j) = \int_0^1 x(u) \varphi_j(u) \, du}$$

Since the λ_j are countable, the WFT is always defined

(DEF) The inverse WFT (iWFT) is defined as:

$$\text{iWFT}(\hat{x}) = \sum_{j \in \mathbb{Z} \setminus \{0\}} \hat{x}(\lambda_j) \varphi_j = x$$

We recover x due to orthonormality of $\{\varphi_j\}_j$.

↳ iWFT is a proper inverse